

Integration by Substitution

Let g be a function whose range is an interval I , and let f be a function that is continuous on I . If g is differentiable on its domain and F is an antiderivative of f on I , then:

$$\int f(g(x))g'(x) dx = F(g(x)) + C$$

If $u = g(x)$ then $du = g'(x)dx$ and

$$\int f(u) du = F(u) + C$$

$$\int f(g(x))g'(x) dx = F(g(x)) + C$$

Outside function Derivative of inside
 ↓ ↘
 Inside function ↑

Example

Find $\int (x^2 + 1)^2 (2x) dx$

Let $g(x) = x^2 + 1$

$$g'(x) = 2x$$

$$f(g(x)) = f(x^2 + 1) = (x^2 + 1)^2 \quad \text{This follows the pattern!!}$$

$$\int (x^2 + 1)^2 (2x) dx$$

$$= \frac{1}{3} (x^2 + 1)^3 + C$$

Example

Find $\int 5 \cos(5x) dx$

Let $g(x) = 5x$

$$g'(x) = 5$$

$$f(g(x)) = f(5x) = \cos(5x) \quad f(x) = \cos(g(x))$$

-Follows the pattern!!

$$\int 5 \cos(5x) dx = \sin(5x) + C$$

Reminder:

$$\int k f(x) dx = k \int f(x) dx$$

-Many integrands contain the essential part (the variable) of $g'(x)$ but are missing the constant multiple.

-We can multiply and divide the constant multiple we need into the problem.

Example

Find $\int x(x^2 + 1)^2 dx$

-This is similar to the previous example but it is missing a factor of 2.

$$= \int (x^2 + 1)^2 \left(\frac{1}{2}\right)(2x) dx$$

$$= \frac{1}{2} \int (x^2 + 1)^2 (2x) dx$$

$$= \frac{1}{2} \left[\frac{(x^2 + 1)^3}{3} \right] + C$$

$$= \frac{1}{6} (x^2 + 1)^3 + C$$

Example-Change of Variables

$$\int f(g(x))g'(x) dx = \int f(u) du = F(u) + C$$

Find $\int \sqrt{2x - 1} dx$

$$u = 2x - 1$$

$$du = 2 dx$$

$$\sqrt{2x - 1} = \sqrt{u} \text{ and } dx = \frac{du}{2}$$

$$\int \sqrt{2x-1} dx$$

$$= \int \sqrt{u} \left(\frac{du}{2} \right)$$

$$= \frac{1}{2} \int u^{1/2} du$$

$$= \frac{1}{2} \left(\frac{u^{3/2}}{3/2} \right) + C$$

$$= \frac{1}{3} u^{3/2} + C$$

$$= \frac{1}{3} (2x-1)^{3/2} + C$$

Example

Find $\int x \sqrt{2x-1} dx$

$$u = 2x - 1$$

$$dx = \frac{du}{2}$$

Since the integrand contains a factor of x , you must solve for x in terms of u .

$$u = 2x - 1$$

$$x = \frac{u+1}{2}$$

Now, substitute:

$$\begin{aligned}
 &= \int \left(\frac{u+1}{2} \right) u^{\frac{1}{2}} \left(\frac{du}{2} \right) \\
 &= \frac{1}{4} \int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}} \right) du \\
 &= \frac{1}{4} \left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} \right) + C \\
 &= \frac{1}{10} (2x-1)^{5/2} + \frac{1}{6} (2x-2)^{3/2} + C
 \end{aligned}$$

Example

$$\int \sin^2(3x) \cos(3x) dx$$

Because $\sin^2(3x) = (\sin(3x))^2$

$$u = \sin(3x)$$

$$du = 3\cos(3x) dx$$

Because $\cos(3x) dx$ is part of the original integral:

$$\frac{du}{3} = \cos(3x) dx$$

Substituting:

$$\int \sin^2(3x) \cos(3x) dx = \int u^2 \frac{du}{3}$$

$$= \frac{1}{3} \int u^2 du$$

$$= \frac{1}{3} \left(\frac{u^3}{3} \right) + C$$

$$= \frac{1}{9} \sin^3(3x) + C$$

Change of Variables-Definite Integrals

Sometimes it is easier to evaluate over bounds of integration for u rather than wait to convert back to x.

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g then,

$$\int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

Example

Evaluate $\int_0^1 x(x^2 + 1)^3 dx$

$$u = x^2 + 1$$

$$du = 2x \, dx$$

Before substituting determine the new upper and lower limits.

Lower:

$$\text{When } x = 0 \quad u = 0^2 + 1 = 1$$

Upper:

$$\text{When } x = 1 \quad u = 1^2 + 1 = 2$$

Now,

$$\int_0^1 x(x^2 + 1)^3 \, dx = \frac{1}{2} \int_0^1 2x(x^2 + 1)^3 \, dx$$

$$= \frac{1}{2} \int_1^2 u^3 \, du$$

$$= \frac{1}{2} \left[\frac{u^4}{4} \right]_1^2$$

$$= \frac{1}{2} \left(4 - \frac{1}{4} \right) = \frac{15}{8}$$

Example

$$\text{Evaluate } A = \int_1^5 \frac{x}{\sqrt{2x-1}} \, dx$$

$$u = \sqrt{2x-1}$$

$$u^2 = 2x - 1$$

$$u^2 + 1 = 2x$$

$$x = \frac{u^2 + 1}{2}$$

$$u \ du = dx$$

Find new limits:

Lower:

$$\text{When } x = 1 \quad u = \sqrt{2-1} = 1$$

Upper:

$$\text{When } x = 5 \quad u = \sqrt{10-1} = 3$$

Substitute:

$$= \int_{1}^{3} \frac{1}{u} \left(\frac{u^2 + 1}{2} \right) u \ du$$

$$= \frac{1}{2} \int_{1}^{3} (u^2 + 1) du$$

$$= \frac{1}{2} \left[\frac{u^3}{3} + u \right]_1^3$$

$$= \frac{1}{2} \left(9 + 3 - \frac{1}{3} - 1 \right)$$

$$= \frac{16}{3}$$